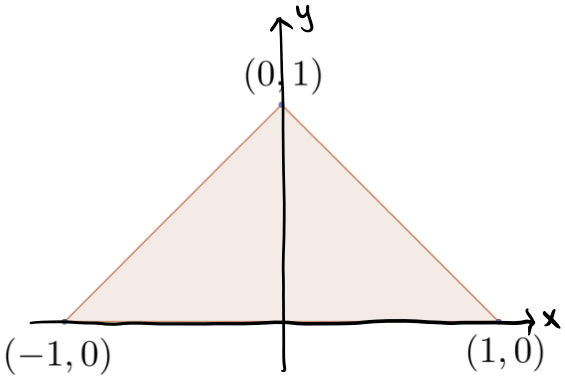
1. Consider the triangle with vertices at $(1,0)$, $(-1,0)$, and $(0,1)$:




a) (2 points) Find the area of the triangle.

The triangle has base 2 and height 1 $\Rightarrow$ Area $= \frac{1}{2}\cdot 2\cdot 1 = \boxed{1}$

b) (4 points) Suppose the density (mass per unit area) is $\rho(x,y) = x^2$. Find the mass of the triangle.

The left side and the right side of the triangle are given by the lines $x = y-1$ and $x = 1-y$.

$\Rightarrow$ The triangular domain $D$ is given by $0 \le y \le 1,\ y-1 \le x \le 1-y$.

$$m = \iint_D \rho(x,y,z)\,dA = \int_0^1 \int_{y-1}^{1-y} x^2\,dx\,dy = \int_0^1 \frac{x^3}{3}\Big|_{x=y-1}^{x=1-y} dy = \int_0^1 \frac{2}{3}(1-y)^3\,dy$$

$$\overset{u=1-y}{=} \int_1^0 \frac{2}{3}u^3\cdot(-1)\,du = -\frac{u^4}{6}\Big|_{u=1}^{u=0} = \boxed{\frac{1}{6}}$$

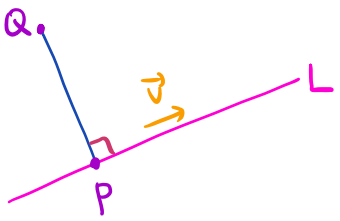
c) (4 points) Suppose the density is $\rho(x,y) = y^2$. Find the mass of the triangle.

$$m = \iint_D \rho(x,y,z)\,dA = \int_0^1 \int_{y-1}^{1-y} y^2\,dx\,dy = \int_0^1 2(1-y)y^2\,dy = \left(\frac{2}{3}y^3 - \frac{1}{2}y^4\right)\Big|_{y=0}^{y=1} = \boxed{\frac{1}{6}}$$

Note In both parts, the density functions are even with respect to $x$. Since the domain $D$ is symmetric about the y-axis, the mass is twice the mass of the part in the first quadrant. Hence you can compute $m = 2\int_0^1 \int_0^{1-y} \rho(x,y,z)\,dx\,dy$.

2. The line $(x, y, z) = (1 - 2t, 2t - 1, t)$, with $t$ as the parameter, is denoted by $L$. The point $Q$ is $(1, 2, 3)$.

a) (5 points) Find a point $P$ on $L$ such that the line from $P$ to $Q$ is orthogonal (perpendicular) to $L$.




$L$ has a direction vector $\vec{v} = (-2, 2, 1)$

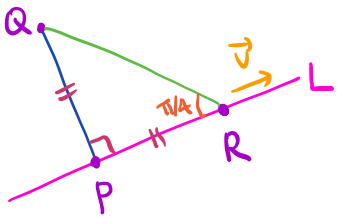
$\overrightarrow{PQ}$ is orthogonal to $\vec{v} \Rightarrow \overrightarrow{PQ} \cdot \vec{v} = 0$

$P = (1-2t, 2t-1, t) \Rightarrow \overrightarrow{PQ} = (2t, 3-2t, 3-t)$.

$\overrightarrow{PQ} \cdot \vec{v} = 0 \Rightarrow (2t, 3-2t, 3-t) \cdot (-2, 2, 1) = 0 \Rightarrow 9 - 9t = 0 \Rightarrow t = 1$.

$\Rightarrow P = (1-2\cdot 1, 2\cdot 1 - 1, 1) = \boxed{(-1, 1, 1)}$

b) (5 points) Find a point $R$ on $L$ such that the line from $R$ to $Q$ makes an angle equal to $45^o$ or $\pi/4$ with $L$.




$\dfrac{|\overrightarrow{PQ}|}{|\overrightarrow{PR}|} = \tan\dfrac{\pi}{4} = 1 \Rightarrow |\overrightarrow{PQ}| = |\overrightarrow{PR}|$

$\overrightarrow{PQ} = (2, 1, 2) \Rightarrow |\overrightarrow{PQ}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

$\overrightarrow{PR}$ is parallel to $\vec{v}$ with $|\overrightarrow{PR}| = |\overrightarrow{PQ}| = 3$.

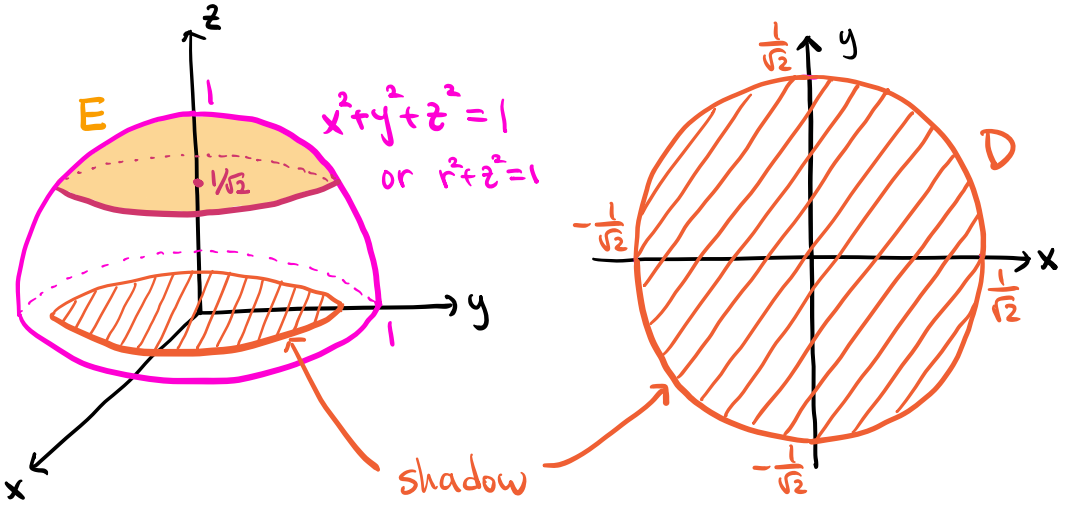
$|\vec{v}| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3 \Rightarrow \overrightarrow{PR} = \pm\vec{v} = \pm(-2, 2, 1)$.

$\Rightarrow R = P \pm \overrightarrow{PR} = (-1, 1, 1) \pm (-2, 2, 1) = \boxed{(-3, 3, 2) \text{ or } (1, -1, 0)}$

Note Alternatively, since $\overrightarrow{QR}$ and $\vec{v}$ form an angle of $\frac{\pi}{4}$ radians, you can set $R = (2u, 3-2u, 3-u)$ from the line equation and use the dot product formula $\overrightarrow{QR} \cdot \vec{v} = |\overrightarrow{QR}||\vec{v}|\cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}|\overrightarrow{QR}||\vec{v}|$ to solve for $u$. This method does not make use of the point $P$, but involves some algebraic computations.

3. Consider the solid sphere $x^2+y^2+z^2 \le 1$. * The answers on the archive are incorrect.

a) (3 points) Find the volume of the sphere above the plane $z = \frac{1}{\sqrt{2}}$.




E : the solid sphere above the plane $z = \frac{1}{\sqrt{2}}$.

In cylindrical coordinates : $x^2+y^2+z^2=1 \Rightarrow r^2+z^2=1 \Rightarrow z=\sqrt{1-r^2}$ $(z \ge 0)$

Intersection : $z=\frac{1}{\sqrt{2}}$ and $z=\sqrt{1-r^2} \Rightarrow \frac{1}{\sqrt{2}}=\sqrt{1-r^2} \Rightarrow \frac{1}{2}=1-r^2 \Rightarrow r=\frac{1}{\sqrt{2}}$.

The shadow on the xy-plane is given by $0 \le \theta \le 2\pi$, $0 \le r \le \frac{1}{\sqrt{2}}$

For each point on the shadow : $\frac{1}{\sqrt{2}} \le z \le \sqrt{1-r^2}$.

$\Rightarrow$ E is given by $0 \le \theta \le 2\pi$, $0 \le r \le \frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}} \le z \le \sqrt{1-r^2}$.

$$\Rightarrow \text{Vol}(E) = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_{1/\sqrt{2}}^{\sqrt{1-r^2}} 1 \cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \left(\sqrt{1-r^2}-\frac{1}{\sqrt{2}}\right) r\,dr\,d\theta$$

(Jacobian)

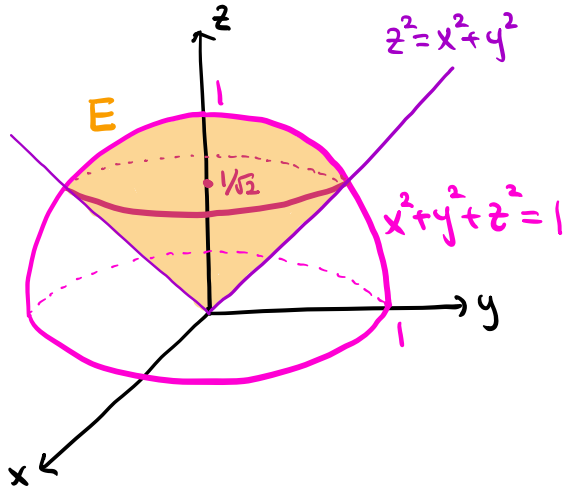
$$= \int_0^{2\pi}\int_1^{1/2} \left(u^{1/2}-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2}\right) du\,d\theta = \int_0^{2\pi} \left(-\frac{u^{3/2}}{3}+\frac{\sqrt{2}}{4}u\right)\Big|_{u=1}^{u=1/2} d\theta$$

($u = 1-r^2$)

$$= \int_0^{2\pi} \frac{1}{3}-\frac{5}{12\sqrt{2}}\,d\theta = \boxed{\frac{2\pi}{3}-\frac{5\pi}{6\sqrt{2}}}$$

**Note** You can also use the bounds $0 \le z \le \frac{1}{\sqrt{2}}$, $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{1-z^2}$ obtained by the cross section method.

b) (3 points) Find the volume inside the sphere and above the cone $z^2 = x^2 + y^2$, $z \geq 0$.




Intersection : $z^2 = x^2+y^2$ and $x^2+y^2+z^2=1$

$\Rightarrow z^2+z^2=1 \Rightarrow z=\frac{1}{\sqrt{2}}$ ($z\geq 0$)

The solid E consists of the two parts:

- $E_a$ is the part above the plane $z=\frac{1}{\sqrt{2}}$
- $E_b$ is the part below the plane $z=\frac{1}{\sqrt{2}}$

The intersection circle is of radius $\frac{1}{\sqrt{2}}$.

$$\text{Vol}(E_a) = \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}} \text{ (a)}, \quad \text{Vol}(E_b) = \underbrace{\frac{1}{3}\cdot\pi\left(\frac{1}{\sqrt{2}}\right)^2\cdot\frac{1}{\sqrt{2}}}_{\text{volume of cone}} = \frac{\pi}{6\sqrt{2}}$$

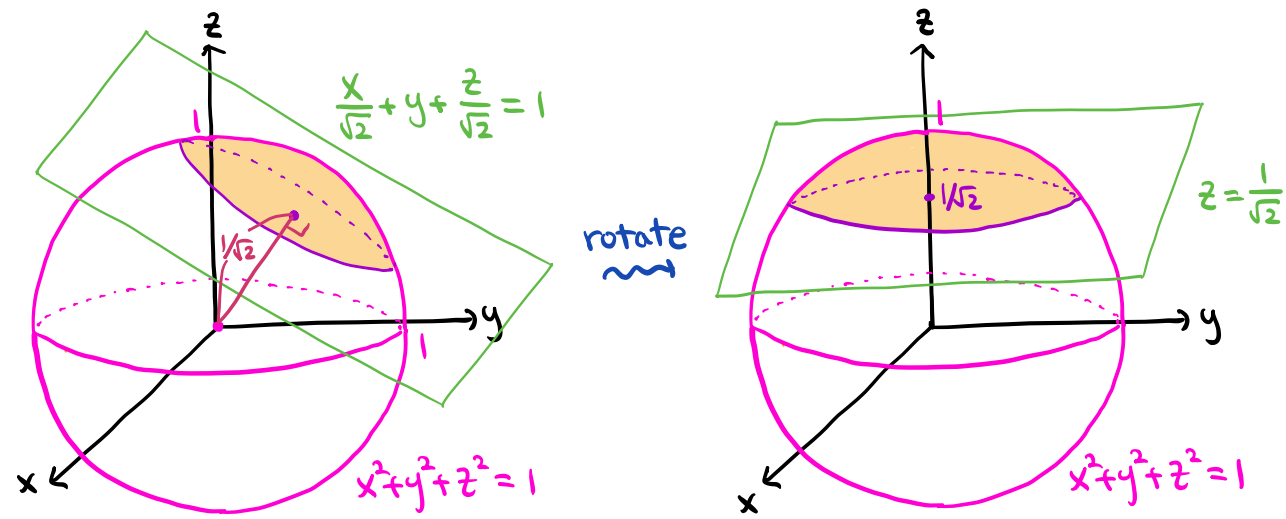
$$\Rightarrow \text{Vol}(E) = \text{Vol}(E_a) + \text{Vol}(E_b) = \frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}} + \frac{\pi}{6\sqrt{2}} = \boxed{\frac{2\pi}{3} - \frac{2\pi}{3\sqrt{2}}}$$

c) (4 points) The plane $\frac{x}{\sqrt{2}} + y + \frac{z}{\sqrt{2}} = 1$ cuts the solid sphere into two pieces. Find the volumes of both the pieces.

The distance from $(0,0,0)$ to the plane $\frac{x}{\sqrt{2}}+y+\frac{z}{\sqrt{2}}-1=0$ is

$$\frac{\left|\frac{0}{\sqrt{2}}+0+\frac{0}{\sqrt{2}}-1\right|}{\sqrt{(1/\sqrt{2})^2+1^2+(1/\sqrt{2})^2}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ We can rotate the plane $\frac{x}{\sqrt{2}}+y+\frac{z}{\sqrt{2}}=1$ to the plane $z=\frac{1}{\sqrt{2}}$.




$E_1$ : the part above the plane $z=\frac{1}{\sqrt{2}} \Rightarrow \text{Vol}(E_1) = \boxed{\frac{2\pi}{3} - \frac{5\pi}{6\sqrt{2}}}$ (a)

$E_2$ : the part below the plane $z=\frac{1}{\sqrt{2}} \Rightarrow \text{Vol}(E_2) = \underbrace{\frac{4\pi}{3}\cdot 1^3}_{\text{volume of sphere}} - \text{Vol}(E_1) = \boxed{\frac{2\pi}{3} + \frac{5\pi}{6\sqrt{2}}}$

4. Consider the surface $2z^2 = x^4 + y^2$.

a) (2 points) Find the gradient of $f(x,y,z) = x^4 + y^2 - 2z^2$ at $(x,y,z) = (1,1,1)$.

$\nabla f = (f_x, f_y, f_z) = (4x^3, 2y, -4z) \Rightarrow \nabla f(1,1,1) = \boxed{(4,2,-4)}$

b) (2 points) Find the tangent plane to the surface at the point $(1,1,1)$.

$2z^2 = x^4+y^2 \Rightarrow x^4+y^2-2z^2 = 0$ : a level surface of $f(x,y,z) = x^4+y^2-2z^2$

$\nabla f(1,1,1) = (4,2,-4)$ is a normal vector

$\Rightarrow$ The tangent plane at $(1,1,1)$ is given by $\boxed{4(x-1)+2(y-1)-4(z-1)=0}$

c) (3 points) Find the area of the tangent plane inside the cylinder $x^2 + y^2 = 1$.

$4(x-1)+2(y-1)-4(z-1)=0 \Rightarrow 4x+2y-4z=2 \Rightarrow z = x+\frac{y}{2}-\frac{1}{2}$.

The surface $S$ is parametrized by $\vec{r}(x,y) = (x, y, x+\frac{y}{2}-\frac{1}{2})$

The domain $D$ is given by $x^2+y^2 \le 1$.

$\text{Area}(S) = \iint_S 1\,dS = \iint_D |\vec{r}_x \times \vec{r}_y|\,dA$

$\vec{r}_x = (1,0,1)$, $\vec{r}_y = (0,1,\frac{1}{2}) \Rightarrow \vec{r}_x \times \vec{r}_y = (-1, -\frac{1}{2}, 1)$

$\Rightarrow \text{Area}(S) = \iint_D \sqrt{(-1)^2+(-\frac{1}{2})^2+1^2}\,dA = \iint_D \frac{3}{2}\,dA = \frac{3}{2}\text{Area}(D) = \frac{3}{2}\cdot \underbrace{\pi \cdot 1^2}_{\text{Area of disk}} = \boxed{\frac{3\pi}{2}}$

d) (3 points) Find the area of the tangent plane inside the elliptical cylinder $\frac{x^2}{9} + \frac{z^2}{16} = 1$.

$4(x-1)+2(y-1)-4(z-1)=0 \Rightarrow 4x+2y-4z=2 \Rightarrow y = -2x+2z+1$.

The surface $S$ is parametrized by $\vec{r}(x,z) = (x, -2x+2z+1, z)$.

The domain $D$ is given by $\frac{x^2}{9} + \frac{z^2}{16} \le 1$.

$\text{Area}(S) = \iint_S 1\,dS = \iint_D |\vec{r}_x \times \vec{r}_z|\,dA$

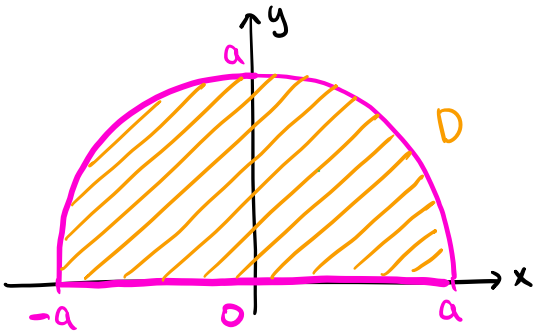
$\vec{r}_x = (1,-2,0)$, $\vec{r}_z = (0,2,1) \Rightarrow \vec{r}_x \times \vec{r}_z = (-2,-1,2)$

$\Rightarrow \text{Area}(S) = \iint_D \sqrt{(-2)^2+(-1)^2+2^2}\,dA = \iint_D 3\,dA = 3\,\text{Area}(D) = 3\cdot \underbrace{\pi \cdot 3 \cdot 4}_{\text{Area of ellipse}} = \boxed{36\pi}$

Note On the exam, you can use the fact that the area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$ is $ab\pi$. For an explanation, see Fact 1 in the Final exam facts note.

5. The density (mass per unit area) is $\rho(x,y) = 1$ throughout this problem.

a) (1 point) Consider the semi-circular plate $x^2 + y^2 \le a^2$ with $y \ge 0$. Find the mass of this plate.




The domain D is a semidisk of radius a.

$$m = \iint_D \rho(x,y)\,dA = \iint_D 1\,dA = \text{Area}(D)$$

$$= \frac{1}{2}\pi a^2 = \boxed{\frac{\pi a^2}{2}}$$

(Area of disk)

b) (4 points) Find the $y$-coordinate of the center of mass of the plate of part (a).

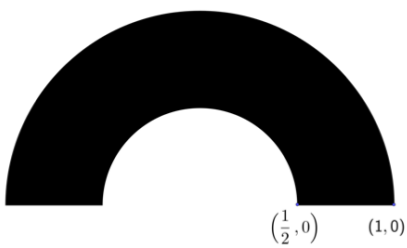
$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{2}{\pi a^2}\iint_D y\,dA$$

In polar coordinates, D is given by $0 \le \theta \le \pi$, $0 \le r \le a$.

$$\Rightarrow \bar{y} = \frac{2}{\pi a^2}\int_0^\pi \int_0^a r\sin\theta \cdot \underbrace{r}_{\text{Jacobian}}\,dr\,d\theta = \frac{2}{\pi a^2}\int_0^\pi \frac{r^3}{3}\sin\theta\Big|_{r=0}^{r=a} d\theta = \frac{2}{\pi a^2}\int_0^\pi \frac{a^3}{3}\sin\theta\,d\theta$$

$$= -\frac{2}{\pi a^2}\cdot\frac{a^3}{3}\cos\theta\Big|_{\theta=0}^{\theta=\pi} = \boxed{\frac{4a}{3\pi}}$$

c) (5 points) Now consider the following half-annular plate (center origin, inner radius $\frac{1}{2}$, outer radius 1):




Find the $y$-coordinate of the center of mass of this plate.

Let D denote the domain.

$$m = \iint_D \rho(x,y)\,dA = \iint_D 1\,dA = \text{Area}(D) = \underbrace{\frac{1}{2}\pi\cdot 1^2}_{\text{outer semidisk}} - \underbrace{\frac{1}{2}\pi\cdot\left(\frac{1}{2}\right)^2}_{\text{inner semidisk}} = \frac{3\pi}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x,y)\,dA = \frac{2}{\pi a^2}\iint_D y\,dA$$

In polar coordinates, the domain D is given by $0 \le \theta \le \pi$, $\frac{1}{2} \le r \le 1$.

$$\Rightarrow \bar{y} = \frac{8}{3\pi}\int_0^\pi \int_{1/2}^1 r\sin\theta \cdot \underbrace{r}_{\text{Jacobian}}\,dr\,d\theta = \frac{8}{3\pi}\int_0^\pi \frac{r^3}{3}\sin\theta\Big|_{r=1/2}^{r=1} d\theta = \frac{8}{3\pi}\int_0^\pi \frac{7}{24}\sin\theta\,d\theta$$

$$= -\frac{8}{3\pi}\cdot\frac{7}{24}\cos\theta\Big|_{\theta=0}^{\theta=\pi} = \boxed{\frac{14}{9\pi}}$$

6. Consider the vector field $\mathbf{F} = (3x^2 + 3y)\mathbf{i} + (3x + 3y^2)\mathbf{j}$.

a) (5 points) Find a potential function $f(x, y)$ such that $\mathbf{F} = \nabla f$.

$\vec{F} = (3x^2+3y,\ 3x+3y^2) \Rightarrow P = 3x^2+3y,\ Q = 3x+3y^2$

$\frac{\partial P}{\partial y} = 3,\ \frac{\partial Q}{\partial x} = 3 \Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \Rightarrow \vec{F}$ is conservative.

$\vec{F} = \nabla f \Rightarrow P = f_x,\ Q = f_y.$

$\int P\,dx = \int 3x^2+3y\,dx = x^3 + 3xy$

$\int Q\,dy = \int 3x+3y^2\,dy = 3xy + y^3$

$\Rightarrow f(x,y) = x^3 + 3xy + y^3$

(To find $f(x,y)$, we collect all terms without duplicates)

Note You get a different potential function by adding a constant.

b) (5 points) Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where the curve $C$ is given by $\mathbf{r}(t) = t^2 \cos t\,\mathbf{i} + t^4 \sin t\,\mathbf{j}$, $0 \le t \le \pi$.

$\int_C \vec{F}\cdot d\vec{r} \overset{\text{Fund. thm}}{=} f(\vec{r}(\pi)) - f(\vec{r}(0))$

$\vec{r}(\pi) = (-\pi^2, 0) \Rightarrow f(\vec{r}(\pi)) = f(-\pi^2, 0) = -\pi^6$

$\vec{r}(0) = (0,0) \Rightarrow f(\vec{r}(0)) = f(0,0) = 0.$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = -\pi^6 - 0 = -\pi^6$

7. Each part of this problem asks you to evaluate a line integral over a closed curve. The curve is always assumed to be counterclockwise.

a) (3 points) Evaluate $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ with $C$ being $x^2+y^2=1$.

This problem considers an integral of the vortex field

$\vec{V}(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

The curve C is a circle centered at (0,0), oriented counterclockwise.

$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \boxed{2\pi}$ (see Fact 3(2) in the Final exam facts note)

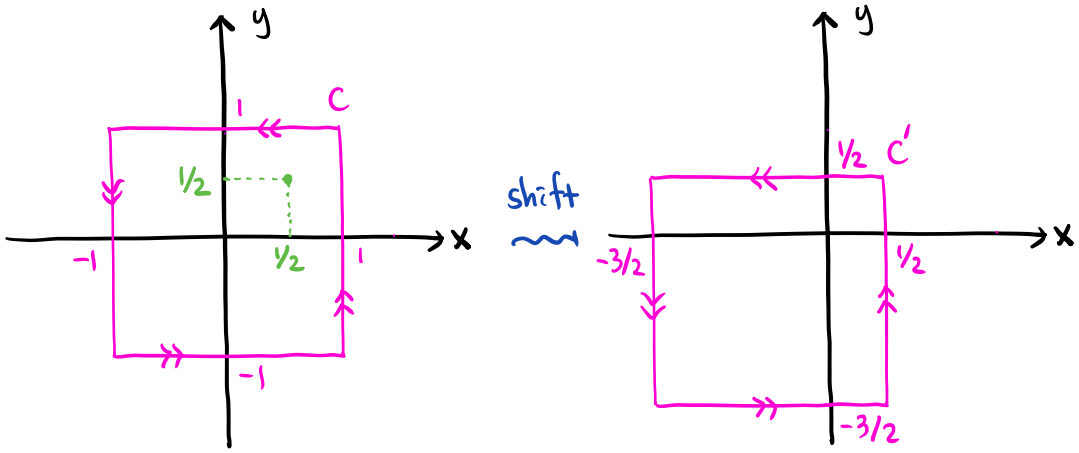
b) (3 points) Evaluate $\int_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ with $C$ being $\frac{(x-1)^2}{2} + (y-2)^2 = 1$.

At (0,0): $\frac{(x-1)^2}{2} + (y-2)^2 = \frac{(0-1)^2}{2} + (0-2)^2 > 1$.

$\Rightarrow$ The curve C does not enclose (0,0)

$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = \boxed{0}$ (see Fact 3(3) in the Final exam facts note)

c) (4 points) Evaluate $\int_c \frac{-2(y-\frac{1}{2})}{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2}dx + \frac{2(x-\frac{1}{2})}{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2}dy$ with $C$ being the square joining the four points $(\pm 1, \pm 1)$.




$\int_C 2\vec{V}(x-\frac{1}{2}, y-\frac{1}{2})\cdot d\vec{r} = \int_{C'} 2\vec{V}(x,y)\cdot d\vec{r} = 2\int_{C'} \vec{V}\cdot d\vec{r}$

C' encloses (0,0) with counterclockwise orientation.

$\Rightarrow \int_{C'} \vec{V}\cdot d\vec{r} = 2\pi$ (see Fact 3(4) in the Final exam facts note)

$\Rightarrow \int_C \vec{V}\cdot d\vec{r} = 2\cdot 2\pi = \boxed{4\pi}$

8. The flux of the vector field $\mathbf{F}$ out of the surface $S$ is given by $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{n}$ is the outward normal. In each part below, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ is the position vector.

a) (2 points) Find the flux of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^2}$ out of the surface $x^2+y^2+z^2=1$.

Let $S$ be the sphere $x^2+y^2+z^2=1$.

The unit normal vector of $S$ is $\vec{n} = (x,y,z) = \vec{r}$.

(See Fact 2 in the Final exam facts note)

$$\Rightarrow \vec{F}\cdot\vec{n} = \frac{\vec{r}}{|\vec{r}|^2}\cdot\vec{r} = \frac{\vec{r}\cdot\vec{r}}{|\vec{r}|^2} = \frac{|\vec{r}|^2}{|\vec{r}|^2} = 1 \text{ on } S.$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS = \iint_S 1\,dS = \text{Area}(S) = 4\pi\cdot 1^2 = \boxed{4\pi}$$

b) (2 points) Find the flux of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$ out of the surface $x^2+y^2+z^2=1$.

$$\vec{r} = (x,y,z) \Rightarrow |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow \vec{F} = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right) : \text{the inverse square field.}$$

The surface $S$ is a sphere centered at $(0,0,0)$, oriented outward.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \boxed{4\pi}$$ (see Fact 4(2) in the Final exam facts note)

Note You can also solve part (b) using the unit normal vector $\vec{n}$.

c) (3 points) Find the flux of $\mathbf{F} = (x-z^2x)\mathbf{i}+(z^2y+y)\mathbf{j}+z\,\mathbf{k}$ out of the surface $x^2+y^2+z^2=1$.

$S$: the sphere $x^2+y^2+z^2=1$ with outward orientation

$E$: the solid ball $x^2+y^2+z^2\le 1$

$\Rightarrow \partial E = S$ is oriented outward.

$$\vec{F} = (x-z^2x, z^2y+y, z) \Rightarrow \text{div}(\vec{F}) = 3$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial E} \vec{F}\cdot d\vec{S} \overset{\text{div. thm}}{=} \iiint_E \text{div}(\vec{F})\,dV = \iiint_E 3\,dV = 3\,\text{Vol}(E) = 3\cdot\frac{4\pi}{3}\cdot 1^3 = \boxed{4\pi}$$

(volume of sphere)

d) (3 points) Find the flux of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$ out of the surface $\left(x-\frac{1}{6}\right)^2+\frac{1}{2}\left(y-\frac{1}{3}\right)^2+\left(z+\frac{1}{5}\right)^2 = 1$.

$\vec{F}$ is the inverse square field as in part (b).

At $(0,0,0)$: $(x-\frac{1}{6})^2+\frac{1}{2}(y-\frac{1}{3})^2+(z+\frac{1}{5})^2 = (0-\frac{1}{6})^2+\frac{1}{2}(0-\frac{1}{3})^2+(0+\frac{1}{5})^2 < 1$.

$\Rightarrow$ The surface $S$ encloses $(0,0,0)$ with outward orientation.

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \boxed{4\pi}$$ (see Fact 4(4) in the Final exam facts note)

9. Let $\mathbf{F} = yz\,\mathbf{i} + 2xz\,\mathbf{j} + 3xy\,\mathbf{k}$ be the vector field for this problem. The surface $S$ is $x^2 + y^2 = 1$ with $-1 \le z \le 1$. The surface is a cylinder. The surface is obviously open at the top or the bottom. As always, the position vector is $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

a) (3 points) The bottom boundary of the surface denoted by $C_1$ is given by $x^2 + y^2 = 1$ and $z = -1$. The orientation is counterclockwise in the $x$-$y$ plane. Find the circulation $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

$C_1$ is parametrized by $\vec{r}_1(t) = (\cos t, \sin t, -1)$ with $0 \le t \le 2\pi$.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1'(t)\,dt$$

$\vec{F} = (yz, 2xz, 3xy) \Rightarrow \vec{F}(\vec{r}_1(t)) = (-\sin t, -2\cos t, 3\cos t \sin t)$.

$\vec{r}_1'(t) = (-\sin t, \cos t, 0) \Rightarrow \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1'(t) = \sin^2 t - 2\cos^2 t$.

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin^2 t - 2\cos^2 t\,dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2} - (1+\cos(2t))\,dt$$

$$= \int_0^{2\pi} -\frac{1+3\cos(2t)}{2}\,dt = -\left(\frac{t}{2} + \frac{3\sin(2t)}{4}\right)\Big|_{t=0}^{t=2\pi} = \boxed{-\pi}$$

b) (2 points) The top boundary of the surface denoted by $C_2$ is given by $x^2 + y^2 = 1$ and $z = 1$. The orientation is counterclockwise in the $x$-$y$ plane. Find the circulation $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

$C_2$ is parametrized by $\vec{r}_2(t) = (\cos t, \sin t, 1)$ with $0 \le t \le 2\pi$.

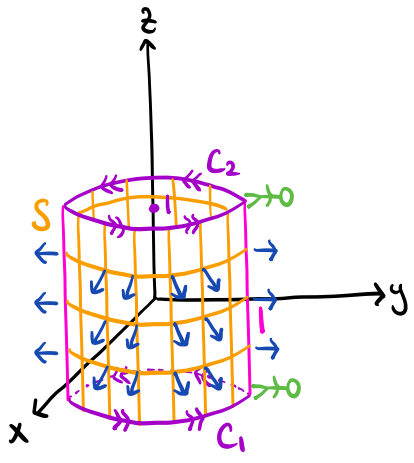
$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_2'(t)\,dt$$

$\vec{F} = (yz, 2xz, 3xy) \Rightarrow \vec{F}(\vec{r}_2(t)) = (\sin t, 2\cos t, 3\cos t \sin t)$.

$\vec{r}_1'(t) = (-\sin t, \cos t, 0) \Rightarrow \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1'(t) = -\sin^2 t + 2\cos^2 t$.

$$\Rightarrow \int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2 t + 2\cos^2 t\,dt = -\int_{C_1} \vec{F} \cdot d\vec{r} \overset{\text{(a)}}{=} \boxed{\pi}$$

c) (5 points) Suppose the unit normal $\mathbf{n}$ to the cylindrical surface $S$ points outward away from the axis of the cylinder. Calculate the flux $\int\int_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS$.




$\partial S = C_1 - C_2$ is positively oriented

($C_2$ is negatively oriented)

*The right hand rule does not apply here.

$$\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} \overset{\text{Stokes' thm}}{=} \int_{\partial S} \vec{F} \cdot d\vec{S} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\overset{\text{(a), (b)}}{=} -\pi - \pi = \boxed{-2\pi}$$